Extremal-point densities of interface fluctuations in a quenched random medium

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We give a number of exact, analytical results for the stochastic dynamics of the density of local extrema (minima and maxima) of linear Langevin equations and solid-on-solid lattice growth models driven by spatially quenched random noise. Such models can describe nonequilibrium surface fluctuations in a spatially quenched random medium, diffusion in a random catalytic environment, and polymers in a random medium. In spite of the nonuniversal character for the quantities studied, their behavior against the variation of the microscopic length scale can present generic features, characteristic of the macroscopic observables of the system.

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I. INTRODUCTION

Recently Toroczkai and co-workers [1–3] studied the dynamics of macroscopically rough surfaces via investigating an intriguing microscopic quantity: the density of extrema (minima) and its finite-size effects. They derived a number of analytical results about these quantities for a large class of nonequilibrium surface fluctuations described by linear Langevin equations, and solid-on-solid (SOS) lattice-growth models. They showed that in spite of the nonuniversal character of the quantities studied, their behavior against the variation of the microscopic length scales can present generic features, characteristic of the macroscopic observables of the system. In addition to surface growth applications, the results can be used to solve the asymptotic scalability problem of massively parallel algorithms for discrete event simulation [2–5], which are extensively used in Monte Carlo–type simulations on parallel architectures. The linear Langevin equations they studied were driven by Gaussian white noise $\eta_i(t)$ at lattice site i at time t, with $\langle \eta_i(t)\eta_j(t')\rangle = 2D\delta_{i,j}\delta(t-t')$, where D is the diffusion constant.

Another interesting type of noise is that of a spatially quenched random noise, with $\langle \eta_i(t)\eta_j(t')\rangle = 2D\delta_{i,j}$. Such noise can describe a growing interface in a quenched random medium, diffusion in a random catalytic environment, and polymers in a random medium [6–10]. Using the technique of Toroczkai [1], we have derived exact analytical results for the density of extremal points for this type of noise. Similar to the case of white noise, we also find generic features characteristic of macroscopic observables of the system. In fact, since our results for the quenched noise are qualitatively similar to those of Gaussian white noise, this shows that the density of local minima is robust with respect to these two types of noises. In Sec. II we will study a discrete lattice model. In Sec. III we will study the continuum model. Section IV is the conclusion.

II. LINEAR MODEL ON THE LATTICE

In this section we focus on the discrete one-dimensional model on the lattice. Let us consider a one-dimensional substrate consisting of L lattice sites, with periodic boundary conditions. For simplicity we take the lattice constant to be unity. We study the discretized linear Langevin equation of the form

$$\partial_t h_i(t) = \nu\nabla^2 h_i(t) - \kappa\nabla^4 h_i(t) + \eta_i(t), \tag{1}$$

where ∇^2 is the discrete Laplacian operator, i.e., $\nabla^2 f_j = f_{j+1} + f_{j-1} - 2f_j$, applied to any lattice function f_j, and $\eta_i(t)$ is a spatially quenched Gaussian with covariance

$$\langle \eta_i(t)\eta_j(t')\rangle = 2D\delta_{i,j}. \tag{2}$$

Stability requires $\nu \geqslant 0$ and $\kappa \geqslant 0$ (as a matter of fact, on the lattice it is enough to have $\nu > 0$ and $\kappa \geqslant -\nu/2$) [11–14]. Starting from a completely flat initial condition, the interface roughens until the correlation length ξ reaches the size of the system $\xi \approx L$, when the roughening saturates over into a steady-state regime. The process of kinetic roughening is controled by the intrinsic length scale $\sqrt{(\kappa/\nu)}$ [11–14]. Below this length scale the roughening is dominated by the surface diffusion or Mullins [15,16] term (the fourth-order operator) but above it, is characterized by the surface tension or Edwards-Wilkinson [17] term.

Introducing the discrete Fourier transform

$$\tilde{h} = \sum_{j=0}^{L-1} e^{-ikj} h_j, \quad k = \frac{2\pi n}{L}, \quad n = 0,1,2,\ldots,L-1. \tag{3}$$

Equation (1) can be transformed into

$$\partial_t \tilde{h}_k(t) = -\{2\nu[1-\cos(k)] + 4\kappa[1-\cos(k)]^2\}\tilde{h}_k(t) + \tilde{\eta}_k(t), \tag{4}$$

with

$$\langle \tilde{\eta}_k(t)\tilde{\eta}_{k'}(t')\rangle = 2DL\,\delta_{(k+k')\mathrm{mod}2\pi,0}. \tag{5}$$

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The equal time structure factor $S(k,t)$ is defined as

$$S^h(k,t)L\,\delta_{(k+k')\mathrm{mod}2\pi,0}\equiv\langle\tilde{h}_k(t)\tilde{h}_{k'}(t)\rangle. \tag{6}$$

Using Eq. (4), we find

$$S(k,t)=2D\left[\frac{1}{\lambda(k)}\right]^2[1-e^{-\lambda(k)t}]^2 \tag{7}$$

where

$$\lambda(k)=[1-\cos(k)]\{2\nu+4\kappa[1-\cos(k)]\}. \tag{8}$$

In the limit $t\to\infty$, we have the steady state-structure function

$$S^h(k)\equiv\lim_{t\to\infty}S^h(k,t)=\frac{2D}{\lambda(k)^2}. \tag{9}$$

We are interested in the density of local minima for the surface described by Eq. (4). The operator which measures this quantity is [1]

$$u=\frac{1}{L}\sum_{i=1}^{L}\Theta(h_{i-1}-h_i)\Theta(h_{i+1}-h_i). \tag{10}$$

In terms of the local slopes, $\phi_i=h_{i+1}-h_i$, the density of local minima is

$$u=\frac{1}{L}\sum_{i=1}^{L}\Theta(-\phi_{i-1})\Theta(\phi_i). \tag{11}$$

Due to translational invariance, the steady-state average of the local minima is $\langle u\rangle=\langle\Theta(-\phi_{i-1})\Theta(\phi_i)\rangle=\langle\Theta(-\phi_1)\Theta(\phi_2)\rangle$. In Ref. [1] it was shown that $\langle u\rangle$ has the form

$$\langle u\rangle=\frac{1}{2\pi}\arccos\left(\frac{\langle\phi_1\phi_2\rangle}{\langle\phi_1{}^2\rangle}\right). \tag{12}$$

We will first find the steady-state structure factor for the slopes. Since

$$\tilde{\phi}_k=(1-e^{-ik})\tilde{h}_k, \tag{13}$$

it follows that $S^\phi(k)=2[1-\cos(k)]S^h(k)$. Therefore we have

$$S^\phi(k)=\frac{4D}{[1-\cos(k)]\{2\nu+4\kappa[1-\cos(k)]\}^2}. \tag{14}$$

This expression is divergent at $k=0$. However, because of the periodic boundary condition $h_L=h_0$, it follows that

$$\tilde{\phi}_{k=0}=\sum_{i=0}^{L-1}\phi_i=h_1-h_0+h_2-h_1+h_3-h_2+\cdots+h_L-h_0=0, \tag{15}$$

$$S^\phi(k=0)=\langle(\tilde{\phi}_{k=0})^2\rangle=0. \tag{16}$$

Of course the equal-time structure factor $S^\phi(k,t)$ given by

$$S^\phi(k,t)=S^\phi(k)[1-e^{-\lambda(k)t}]^2 \tag{17}$$

is finite for all finite t.

The slope-slope correlation function is given by

$$C_L^\phi(l)\equiv\langle\phi_i\phi_{i+l}\rangle=\frac{1}{L}\sum_{n=1}^{L-1}e^{i(2\pi n/L)l}S^\phi\left(\frac{2\pi n}{L}\right), \tag{18}$$

where the summation starts at $n=1$, since the $n=0$ term is zero. This function can be calculated using the Poisson summation formula [18]

$$\sum_{\alpha}^{\beta}f(n)=\frac{1}{2}[f(\alpha)+f(\beta)]+\int_\alpha^\beta f(x)dx+2\sum_{m=1}^{\infty}\int_\alpha^\beta dx\,f(x)\cos(2\pi mx). \tag{19}$$

We find that

$$C_L^\phi(l)=\frac{D}{\pi}\Bigg\{g_2(l)+lg_3(l)+\frac{b^L}{1-b^L}[g_2(l)+g_2(-l)]+L\frac{b^L}{(1-b^L)^2}[g_3(l)+g_3(-l)]+l\frac{b^L}{1-b^L}[g_3(l)-g_3(-l)]\Bigg\}, \tag{20}$$

where

$$g_2(l)=\frac{\pi}{2\kappa}b^l\left[\frac{2\nu(\nu+4\kappa)-\sqrt{\nu}(\nu+4\kappa)-\nu^{3/2}}{\nu^{5/2}\sqrt{\nu}(\nu+4\kappa)^{3/2}}\right], \tag{21}$$

$$g_3(l)=-\frac{\pi}{\kappa}\frac{b^l}{\nu^3(\nu+4\kappa)}, \tag{22}$$

$$b=1-\frac{\sqrt{\nu(\nu+4\kappa)}-\nu}{2\kappa}<1. \tag{23}$$

In the limit $L\to\infty$, we can neglect the terms that vanish exponentially with L and we have

$$C_L^\phi(l)=\frac{D}{\pi}[g_2(l)+lg_3(l)]. \tag{24}$$

Substituting the expressions for $g_2(l)$ and $g_3(l)$, we find

$$\frac{C_L^\phi(1)}{C_L^\phi(0)}=\frac{b}{1-\nu^{3/2}}=\frac{\nu+2\kappa-\sqrt{\nu(\nu+4\kappa)}}{2\kappa(1-\nu^{3/2})}. \tag{25}$$

Therefore the average density of local minima is given by

$$\langle u\rangle=\frac{1}{2\pi}\arccos\left(\frac{\nu+2\kappa-\sqrt{\nu(\nu+4\kappa)}}{2\kappa(1-\nu^{3/2})}\right). \tag{26}$$

We can see that the average density of minina vanishes for $\nu=0$, i.e., for the pure Mullins or diffusion case. But for finite ν, the average density of minina is finite. The pure Edwards-Wilkinson or surface tension case can be obtained by taking the limit $\kappa\to0$ in the last expression. It gives $\langle u\rangle=\arccos(0)/(2\pi)=1/4$. In order to check this result one can go back to the slope-slope correlation

$$C_L^{\phi}(l)=\frac{2D}{\nu^2 L}\sum_{n=1}^{L-1}\frac{e^{i(2\pi n/L)l}}{\left\{2\left[1-\cos\left(\frac{2\pi n}{L}\right)\right]\right\}}. \quad (27)$$

Using the Poisson summation formula again we find

$$C_L^{\phi}(l)=\frac{D}{\nu^2}(1-l). \quad (28)$$

Therefore, for the pure Edwards-Wilkinson case, the average density of local minima is

$$\langle u\rangle=\frac{1}{2\pi}\arccos\left[\frac{C_L^{\phi}(1)}{C_L^{\phi}(0)}\right]=\frac{1}{2\pi}\arccos(0)=\frac{1}{4}. \quad (29)$$

This agrees with the result obtained by taking the limit $\kappa\to 0$ in Eq. (26).

III. EXTREMAL-POINT DENSITIES ON THE CONTINUUM

We are interested in the following type of linear stochastic equations:

$$\frac{\partial h}{\partial t}=-\nu(-\nabla^2)^{z/2}h+\eta(x), \quad (30)$$

where ν, $z>0$ and $x\in[0,L]$ with initial condition $h(x,0)=0$ for all x. The Edwards-Wilkinson and Mullins cases correspond to $z=2$ and $z=4$, respectively. $\eta(x)$ is a time-independent quenched noise term with zero mean $\langle\eta(x)\rangle=0$, and covariance

$$\langle\eta(x)\eta(x')\rangle=2D\,\delta(x-x'). \quad (31)$$

We have also chosen periodic boundary conditions $h(x+nL,t)=h(x,t)$ and $\eta(x+nL)=\eta(x,t)$ for all integer n. Introducing the Fourier transform where $k=2\pi n/L$, $n=\ldots,-2,-1,0,1,2,\ldots$,

$$f(x)=\sum_k \tilde{f}(k)e^{ikx},\quad \tilde{f}(k)=\frac{1}{L}\int_{-L}^{L}dx\, f(x)e^{-ikx}, \quad (32)$$

the solution for h has the form

$$\tilde{h}(k,t)=\int_0^t dt'\,\exp[-\nu|k|^z(t-t')]\tilde{\eta}(k), \quad (33)$$

$$\langle\eta(k)\eta(k')\rangle=\frac{2D}{L}\delta_{k,-k'}. \quad (34)$$

Using these equations, the equal-time, two-point correlation can be calculated as

$$\langle\tilde{h}(k,t)\tilde{h}(k',t)\rangle=\frac{2D}{L}\frac{\delta_{k,-k'}}{\nu^2|k|^{2z}}[1-\exp(-\nu|k|^z t)]^2$$
$$=S(k,t)\,\delta_{k,k'}, \quad (35)$$

where $S(k,t)$ is the structure function given by

$$S(k,t)=\frac{2D}{L}\frac{1}{\nu^2|k|^{2z}}[1-\exp(-\nu|k|^z t)]^2. \quad (36)$$

The extremal-point density is defined through the functions

$$C_q(L,t)=\left\langle\frac{1}{L}\int_0^L dx\left|\frac{\partial^2 h}{\partial x^2}\right|^{q+1}\delta\left(\frac{\partial h}{\partial x}\right)\right\rangle, \quad (37)$$

$$U_q(L,t)=\left\langle\frac{1}{L}\int_0^L dx\left(\frac{\partial^2 h}{\partial x^2}\right)^{q+1}\delta\left(\frac{\partial h}{\partial x}\right)\Theta\left(\frac{\partial^2 h}{\partial x^2}\right)\right\rangle, \quad (38)$$

where $\Theta(x)$ is the Heavyside step function and $q>0$ can be conceived as an inverse temperature. The limit $q\to 0^+$ gives the stochastic average of the density of nondegenerate extrema and minima,

$$\overline{C}(L,t)=\lim_{q\to 0}C_q(L,t),\quad \overline{U}(L,t)=\lim_{q\to 0}U_q(L,t), \quad (39)$$

and the limit $q=1$ gives the stochastic averages of the mean curvature at extrema and

$$\overline{K}_{\rm ext}=\frac{C_1(L,t)}{\overline{C}(L,t)},\quad \overline{K}_{\rm min}=\frac{U_1(L,t)}{\overline{U}(L,t)} \quad (40)$$

These functions can be written in terms of the function $\phi_m(L,t)$ defined as

$$\phi_m(L,t)=\sum_k |k|^m S(k,t), \quad (41)$$

$$U_q(L,t)=\frac{2^{(q-2)/2}}{\pi}\Gamma\left(\frac{q}{2}+1\right)\frac{[\phi_4(L,t)]^{(q+2)/2}}{\sqrt{\phi_2(L,t)}}, \quad (42)$$

$$C_q(L,t)=2U_q(L,t), \quad (43)$$

$$\overline{U}(L,t)=\frac{1}{2\pi}\sqrt{\frac{\phi_4(L,t)}{\phi_2(L,t)}}, \quad (44)$$

$$\overline{K}(L,t)=\overline{K}_{\rm min}(L,t)=\sqrt{\frac{\pi}{2}}\sqrt{\phi_4(L,t)}. \quad (45)$$

Substituting the expression for $S(k,t)$, the function $\phi_m(L,t)$ can be calculated with the result

$$\phi_m(L,t)=\frac{2D}{\nu^2 L}\sum_{n=0}^{L/2a}\left(\frac{2\pi n}{L}\right)^{m-2z}\{1-\exp[-(\xi^2 2\pi n/L)^z]\}^2, \quad (46)$$

where a is the lattice spacing and $\xi=(\nu t)^{1/z}$ is the correlation length. In the future we will drop the $n=0$ term from Eq. (46) since it is zero for all finite ξ.

A. Steady-state regime

For $t\to\infty$ or $\xi\to\infty$ the steady-state functions are

$$\phi_m(L,\infty)=\frac{2D}{\nu^2 L}\left(\frac{2\pi}{L}\right)^{m-2z}\sum_{n=1}^{L/2a}n^{m-2z}. \quad (47)$$

The summation can be carried out. The critical values of z are given by $m-2z=-1$: $z=\frac{3}{2}$ for $m=2$ and $z=\frac{5}{2}$ for $m=4$. We have to consider the following five separate cases.

(i) $z>\frac{5}{2}$. This case includes the case of the pure Mullins term with $z=4$. In this case all quantities are convergent as $a\to 0^+$ and we have

$$U_q(L,\infty)=\left(\frac{4D}{\nu^2}\right)^{q/2}\Gamma\left(\frac{q}{2}+1\right)\times(2\pi)^{-(z-2)q}L^{-q[(5/2)-z]-1}\frac{[\zeta(2z-4)]^{(q+1)/2}}{[\zeta(2z-2)]^{1/2}}, \tag{48}$$

$$\overline{U}(L,\infty)=\frac{1}{L}\sqrt{\frac{\zeta(2z-4)}{\zeta(2z-2)}}, \tag{49}$$

$$\overline{K}(L,\infty)=\nu^{-1}(2\pi)^{-(z-2)}L^{z-(5/2)}\sqrt{\pi D\zeta(2z-4)}, \tag{50}$$

where $\zeta(x)$ is the Rieman-zeta function. This shows that the density of minima goes to zero as 1/L. Therefore for the pure Mullins term, with $z=4$, the density of local minina vanishes for large L in the steady state. The mean curvature diverges as $L^{(2z-5)/2}$.

(ii) $z=\frac{5}{2}$. In this case ϕ_4 is marginal while ϕ_2 is still convergent as $a\to 0^+$ and we have

$$U_q(L,\infty)=\left(\frac{4D}{\nu^2}\right)^{q/2}\Gamma\left(\frac{q}{2}+1\right)\times(2\pi)^{-q/2}\frac{1}{L}\frac{\left[\ln\frac{L}{2a}+C\right]^{(q+1)/2}}{\sqrt{\zeta(3)}}, \tag{51}$$

$$\overline{U}(L,\infty)=\frac{1}{L}\sqrt{\frac{\ln\frac{L}{2a}+C}{\zeta(3)}}, \tag{52}$$

$$\overline{K}(L,\infty)=\sqrt{\frac{2D}{\nu^2}}\sqrt{\ln\frac{L}{2a}+C}. \tag{53}$$

In this case the density of minima goes to zero as $1/L$, but with a logarithmic correction. The mean curvature diverges logarithmically.

(iii) $\frac{3}{2}<z<\frac{5}{2}$. This case includes the case of the pure Edwards-Wilkinson term, with $z=2$. In this case ϕ_2 is still convergent as $a\to 0^+$, but ϕ_4 is divergent in this limit,

$$\phi_4(L,\infty)=\frac{D}{\pi\nu^2(5-2z)}\left(\frac{\pi}{a}\right)^{5-2z}, \tag{54}$$

$$U_q(L,\infty)=\left(\frac{4D}{\nu^2(5-2z)}\right)^{q/2}\Gamma\left(\frac{q}{2}+1\right)(2\pi)^{-[2+(q+1/2)-z]}\times\left(\frac{\pi}{a}\right)^{(5-2z)[(q+1)/2]}\frac{L^{-[z-(3/2)]}}{\sqrt{(5-2z)\zeta(2z-2)}}, \tag{55}$$

$$\overline{U}(L,\infty)=(2\pi)^{-[(5/2)-z]}\left(\frac{\pi}{a}\right)^{(5-2z)/2}\frac{L^{-[z-(3/2)]}}{\sqrt{(5-2z)\zeta(2z-2)}}. \tag{56}$$

For $z=2$, the last expression gives

$$\overline{U}(L,\infty)=\frac{1}{\sqrt{2\zeta(2)}}\frac{1}{\sqrt{La}}\sim\frac{1}{\sqrt{2\zeta(2)}},\quad z=2$$

for $La\approx$const. Therefore, for the pure Edwards-Wilkinson case $z=2$ and the density of local minima is a constant, in agreement with the result of the discrete lattice model, Eqs. (26) and (29). The density of minima vanishes with the system size with exponent $(2z-3)/2$. Also the dependence of U_q on L now decouples from q. The mean curvature is still given by the expression (53) of case (ii).

(iv) $z=\frac{3}{2}$. In this case both ϕ_2 and ϕ_4 diverge as $a\to 0$, but ϕ_2 diverges only logarithmically. By setting $z=\frac{3}{2}$ in Eq. (55) in case (iii) and replacing the Rieman-zeta function ζ by $[\ln(L/2a)+C]$, one has

$$U_q(L,\infty)=\left(\frac{D}{\pi}\right)^{q/2}\Gamma\left(\frac{q}{2}+1\right)\left(\frac{\pi}{a}\right)^{q+1}\frac{1}{\sqrt{2}\sqrt{\ln\left(\frac{L}{2a}\right)+C}}, \tag{57}$$

$$\overline{U}(L,\infty)=\left(\frac{\pi}{a}\right)\frac{1}{\sqrt{2}\sqrt{\ln\left(\frac{L}{2a}\right)+C}}. \tag{58}$$

The density of minima vanishes logarthmically with L. The dependence of U_q on L decouples from q. The mean curvature is the same as that of case (ii) since it depends only on ϕ_4.

(v) $1<z<\frac{3}{2}$. In this case both ϕ_2 and ϕ_4 are divergent as $a\to 0$,

$$U_q(L,\infty)=\left(\frac{4D}{\nu^2}\right)^{q/2}\Gamma\left(\frac{q}{2}+1\right)\times(2\pi)^{-(q/2)-1}\left(\frac{\pi}{a}\right)^{(5-2z)(q/2)+1}\frac{\sqrt{3-2z}}{(5-2z)^{(q+1)/2}}, \tag{59}$$

$$\overline{U}(L,\infty)=\frac{1}{2a}\sqrt{\frac{3-2z}{5-2z}}. \tag{60}$$

The density of the minima is a constant, independent of the system size. The mean curvature is still given by Eq. (53) of case (ii). Only in this case the density of local minima diverges as $a\to 0$.

B. Scaling regime

For finite time, the function $\phi_m(L,t)$ defined in Eq. (46) can be calculated by using the Poisson summation formula. The result is

$$\phi_m(L,t)\approx\frac{D}{\pi\nu^2}\frac{1}{m-2z+1}\left(\frac{\pi}{a}\right)^{m-2z+1}-\frac{D}{\pi\nu^2}\xi^{-(m-2z+1)}\frac{2}{z}(1-2^{-(m-z+1)/z})\times\Gamma\left(\frac{m-2+z}{z}\right)\left[1-E_m\left(\frac{L}{\xi},\frac{\xi}{\alpha}\right)\right] \quad (61)$$

for $z<(m+1)/2$, where

$$E_m\left(\frac{L}{\xi},\frac{\xi}{a}\right)=\frac{z}{(1-2^{-(m-z+1)/z})\Gamma\left(\frac{m-2z+1}{z}\right)}\times\sum_{n=1}^{\infty}\int_0^{\pi\xi/a}x^{m-2z}\cos(Lnx/\xi)(1-e^{-x^z})^2dx. \quad (62)$$

Using the property of the Γ function, this can be written as

$$E_m\left(\frac{L}{\xi},\frac{\xi}{a}\right)=\frac{(2z-m-1)(z-m-1)}{z(1-2^{-(m-z+1)/z})\Gamma\left(\frac{m+1}{z}\right)}\times\sum_{n=1}^{\infty}\int_0^{\pi\xi/a}x^{m-2z}\cos(Lnx/\xi)(1-e^{-x^z})^2dx. \quad (63)$$

The oscillating terms in E_m will give finite-size corrections, as long as $L/\xi\gg1$.

The case $2z=m+1$ can also be calculated with the result

$$\int_0^{\pi\xi/a}\frac{dx}{x}(1-e^{-x^z})^2=\ln\left(\frac{\pi\xi}{a}\right)-\frac{2}{z}\mathrm{Ei}\left[-\left(\frac{\pi\xi}{a}\right)^z\right]+\frac{1}{z}\mathrm{Ei}\left[-2\left(\frac{\pi\xi}{a}\right)^z\right]+\frac{C}{z}, \quad (64)$$

where $\mathrm{Ei}(x)$ is the exponential integral function and C is a constant. For large x, the exponential integral function vanishes exponentially fast [19]: $\mathrm{Ei}(x)\sim-e^{-x}/x$, and can be neglected. Therefore we have

$$\phi_m(L,t)\approx\frac{D}{\pi\nu^2}\left[\ln\left(\frac{\pi\xi}{a}\right)+\frac{C}{z}\right]+\frac{D}{\pi\nu^2}F_m\left(\frac{L}{\xi},\frac{\xi}{a}\right),\quad z=\frac{m+1}{2} \quad (65)$$

where

$$F_m\left(\frac{L}{\xi},\frac{\xi}{a}\right)=2\sum_{n=1}^{\infty}\int_0^{\pi\xi/a}x^{m-2z}\cos(Lnx/\xi)(1-e^{-x^z})^2dx. \quad (66)$$

Thus in the scaling limit, the temporal behavior of $\phi_m(L,t)$ is a logarithmic time dependence plus a constant, as long as $L/\xi\gg1$.

Just as in the steady-state case, we have to distinguish the five cases depending on the value of z with respect to the critical values $\frac{3}{2}$ and $\frac{5}{2}$. For the sake of simplicity of writing, we will sometimes omit the arguments of E_m and F_m.

(i) $z>\frac{5}{2}$. This case includes the case of the pure Mullins term. In this case we have

$$\phi_m(L,t)=\frac{D}{\pi\nu^2}\frac{2z\xi^{2z-m-1}}{(2z-m-1)(m+1-z)}\times(1-2^{(m+1-z)/z})\Gamma\left(\frac{m+1}{z}\right), \quad (67)$$

$$U_q(L,t)=\frac{\Gamma\left(\frac{q}{2}+1\right)}{2\pi}\left(\frac{Dz}{\pi\nu^2}\right)^{q/2}\xi^{(2z-5)(q/2)-1}\times\left[4(1-2^{(z-5)/z})\frac{\Gamma\left(\frac{5}{z}\right)}{(2z-5)(5-z)(1-E_4)}\right]^{(q+1)/2}\times\left[\frac{(2z-3)(3-z)}{2\Gamma\left(\frac{3}{2}\right)(1-2^{(z-3)/z})(1-E_2)}\right]^{1/2}, \quad (68)$$

$$\overline{U}(L,t)=\frac{1}{2\pi\xi}\left[\frac{2(1-2^{(z-5)/z})\Gamma\left(\frac{5}{z}\right)(2z-3)(3-z)(1-E_4)}{(1-2^{(z-3)/z})\Gamma\left(\frac{3}{z}\right)(2z-5)(5-z)(1-E_2)}\right]^{1/2}, \quad (69)$$

$$\overline{K}(L,t)=\sqrt{\frac{Dz\Gamma\left(\frac{5}{z}\right)(1-2^{(z-5)/z})(1-E_4)}{\nu^2(2z-5)(5-z)}}\,\xi^{z-(5/2)}. \quad (70)$$

Therefore in this case $U_q(L,t)\sim t^{-[2-q(2z-5)]/z}$, $U(L,t)\sim t^{-1/z}$, and $K(L,t)\sim t^{(2z-5)/z}$ for $L/\xi\gg1$. In the Mullins case, with $z=4$, the density of local minima decreases in time as $t^{-1/2}$ and vanishes after very long time.

(ii) $z=\frac{5}{2}$. In this case ϕ_2 is still given as in case (i) but ϕ_4 is given by

$$\phi_4(L,t)=\frac{D}{\pi\nu^2}\left(\ln\frac{\pi\xi}{a}+\frac{C}{z}\right)+\frac{D}{\pi\nu^2}F_4\left(\frac{L}{\xi},\frac{\xi}{a}\right), \quad (71)$$

$$U_q(L,t)=\frac{\Gamma\left(\frac{q}{2}+1\right)}{2\pi\xi}\left(\frac{2D}{\pi\nu^2}\right)^{q/2}\left[\ln\frac{\pi\xi}{a}\right]^{(q+1)/2}\times\frac{\left\{1+\left[\ln\frac{\pi\xi}{a}\right]^{-1}\left(\frac{2C}{5}+F_4\right)\right\}^{(q+1)/2}}{\sqrt{2(1-2^{-1/5})\Gamma\left(\frac{6}{5}\right)(1-E_2)}}, \quad (72)$$

$$\overline{U}(L,t)=\frac{1}{2\pi\xi}\left[\ln\frac{\pi\xi}{a}\right]^{1/2}\frac{\left\{1+\left[\ln\frac{\pi\xi}{a}\right]^{-1}\left(\frac{2C}{5}+F_4\right)\right\}^{1/2}}{\sqrt{2(1-2^{-1/5})\Gamma\left(\frac{6}{5}\right)(1-E_2)}},\tag{73}$$

$$\overline{K}(L,t)=\sqrt{\frac{D}{2\nu^2}\left(\ln\frac{\pi\xi}{a}\right)\left\{1+\left[\ln\frac{\pi\xi}{a}\right]^{-1}\left(\frac{2C}{5}+F_4\right)\right\}}.\tag{74}$$

One can observe the logarithmic correction for the quantities U_q, U, and K.

(iii) $\frac{3}{2}<z<\frac{5}{2}$. This case includes the case of the pure Edwards-Wilkinson term with $z=2$,

$$\phi_2(L,t)=\frac{2D}{\pi\nu^2}\frac{1-2^{(z-3)/z}}{(2z-3)(3-z)}\Gamma\left(\frac{3}{z}\right)(1-E_2)\xi^{2z-3},\tag{75}$$

$$\phi_4(L,t)=\frac{D}{\pi\nu^2}\frac{1}{5-2z}\left(\frac{\pi}{a}\right)^{5-2z}-\frac{2D}{\pi\nu^2 z}\times[1-2^{-(5-z)z}]\xi^{-(5-2z)}\Gamma\left(\frac{5-2z}{z}\right)(1-E_4),\tag{76}$$

$$U_q(L,t)=\frac{1}{2\pi}\Gamma\left(\frac{q}{2}+1\right)\left(\frac{2D}{\pi\nu^2}\right)^{q/2}(5-2z)^{[(q+1)/2]}\left(\frac{\pi}{a}\right)^{-[(q+1)/2](5-2z)}\xi^{-(2z-3)/2}\left[\frac{(2z-3)(3-z)}{2(1-2^{-(3-z)/z})\Gamma(3/z)(1-E_2)}\right]^{1/2}\times\left[1-\frac{2}{z}\left(\frac{a}{\pi\xi}\right)^{5-2z}(5-2z)[1-2^{-(5-z)z}]\Gamma\left(\frac{5-2z}{z}\right)(1-E_4)\right]^{(q+1)/2},\tag{77}$$

$$\overline{U}(L,t)=\frac{1}{2\pi}\left[\frac{(2z-3)(3-z)}{2(5-2z)(1-2^{-(3-z)/z})\Gamma\left(\frac{3}{z}\right)}\right]^{1/2}\left(\frac{\pi}{a}\right)^{(5-2z)/2}\xi^{-[(2z-3)/2]}\otimes\times\left[1-\left(\frac{a}{\pi\xi}\right)^{5-2z}\frac{2}{z}(5-2z)(1-2^{-(5-z)/z})\Gamma\left(\frac{5-2z}{z}\right)(1-E_4)\right]^{1/2}(1-E_2)^{-1/2},\tag{78}$$

$$\overline{K}(L,t)=\sqrt{\frac{D}{2\nu^2(5-2z)}}\left(\frac{\pi}{a}\right)^{(5-2z)/z}\left[1-\frac{2}{z}\Gamma\left(\frac{5-2z}{z}\right)\left(\frac{a}{\pi\xi}\right)^{5-2z}[1-2^{-(5-z)/z}](5-2z)(1-E_4)\right]^{1/2}.\tag{79}$$

From Eq. (78) one can see that in the case of the pure Edwards-Wilkinson term, with $z=2$, the density of local minima goes as $(\xi a)^{-1/2}$. After a long time, $\xi\to\infty$, $a\to0$, with $\xi a\to$cons. This agrees with the steady-state and discrete lattice cases.

(iv) $z=\frac{3}{2}$,

$$\phi_2(L,t)=\frac{D}{\pi\nu^2}\left[\ln\frac{\pi\xi}{a}+\frac{2C}{3}+F_2\right],\tag{80}$$

$$\phi_4(L,t)=\frac{D}{2\pi\nu^2}\left(\frac{\pi}{a}\right)^2-\frac{2D}{\pi\nu^2 z\xi^2}(1-2^{-4/3})\Gamma\left(\frac{4}{3}\right)(1-E_4),\tag{81}$$

$$U_q(L,t)=\frac{\Gamma\left(\frac{q}{2}+1\right)}{4\pi}\left(\frac{D}{\pi\nu^2}\right)^{q/2}\left(\frac{\pi}{a}\right)^{q+1}\left[\frac{\pi\xi}{a}\right]^{-1/2}\frac{\left[1-\frac{8}{3}\Gamma\left(\frac{4}{3}\right)\left(\frac{a}{\pi\xi}\right)^2(1-2^{-4/3})(1-E_4)\right]^{(q+1)/2}}{\sqrt{1+\left[\ln\frac{\pi\xi}{a}\right]^{-1}\left(\frac{2C}{3}+F_2\right)}},\tag{82}$$

$$\overline{U}(L,t)=\frac{1}{4a}\left[\ln\frac{\pi\xi}{a}\right]^{-1/2}\frac{\left[1-\frac{8}{3}\Gamma\left(\frac{4}{3}\right)\left(\frac{a}{\pi\xi}\right)^2(1-2^{-4/3})(1-E_4)\right]^{(q+1)/2}}{\sqrt{1+\left[\ln\frac{\pi\xi}{a}\right]^{-1}\left(\frac{2C}{3}+F_2\right)}},\tag{83}$$

$K(L,t)$ is given by the same equation as in case (iii).

(v) $1<z<\frac{3}{2}$. In this case ϕ_4 is given by Eq. (76) of case (iii) and ϕ_2 is given by

$$\phi_2(L,t)=\frac{D}{\pi\nu^2}\frac{1}{3-2z}\left(\frac{\pi}{a}\right)^{3-2z}-\frac{2D}{\pi\nu^2 z}[1-2^{-(3-z)z}]\xi^{-(3-2z)}\Gamma\left(\frac{3-2z}{z}\right)(1-E_2), \tag{84}$$

$$U_q(L,t)=\frac{\Gamma\left(\frac{q}{2}+1\right)}{2\pi}\left(\frac{2D}{\pi\nu^2}\right)^{q/2}\sqrt{\frac{3-2z}{(5-2z)^{q+1}}}\left(\frac{\pi}{a}\right)^{(5-2z)(q/2)+1}\frac{\left[1-\left(\frac{a}{\pi\xi}\right)^{5-2z}\frac{2z}{5-z}(1-2^{-(5-z)/z})\Gamma\left(\frac{5}{z}\right)(1-E_4)\right]^{(q+1)/2}}{\sqrt{1-\left(\frac{a}{\pi\xi}\right)^{3-2z}\frac{2z}{3-z}(1-2^{-(3-z)/z})\Gamma\left(\frac{3}{z}\right)(1-E_2)}}, \tag{85}$$

$$\overline{U}(L,t)=\frac{1}{2a}\sqrt{\frac{3\text{-}2z}{5-2z}}\frac{\sqrt{1-\left(\frac{a}{\pi\xi}\right)^{5-2z}\frac{2z}{5-z}(1-2^{-(5-z)/z})\Gamma\left(\frac{5}{z}\right)(1-E_4)}}{\sqrt{1-\left(\frac{a}{\pi\xi}\right)^{3-2z}\frac{2z}{3-z}(1-2^{-(3-z)/z})\Gamma\left(\frac{3}{z}\right)(1-E_2)}}.$$

IV. CONCLUSION

We have studied the density of local minima in linear Langevin equations driven by spatially quenched random noise. In spite of the nonuniversal character of the quantities studied, their behavior against the variation of the microscopic length scales can present generic features. We find that the density of local minima vanishes in the limit of a large system size in the case of the pure Mullins term but is finite in the case of the pure Edwards-Wilkinson term, or when both term are present. For length scales larger than $\sqrt{(\kappa/\nu)}$, the roughening is dominated by the Edwards-Wilkinson term [17]. Therefore it follows that in the steady state, for large system size, the density of local minima is always given by the isotropic value $\langle u\rangle=\frac{1}{4}$, as long as ν is not strictly zero. Also, since the results of the quenched random noise is qualitatively similar to those of the Gaussian white noise [1], this shows that the density of local minima is robust with respect to these two types of noises.

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